Higher-order Osculating Eigenvectors of Symmetric Tensors

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We work in the space of real n-ary symmetric tensors of order d, or of real homogeneous polynomials in n variables of degree d. We equip this space with the Bombieri-Weyl inner product $\langle\cdot,\cdot\rangle_{\mathrm{BW}}$. Let X be the affine cone of real symmetric tensors of rank at most one. For a fixed real symmetric tensor f, we study the (squared) distance function from f restricted to X induced by the Bombieri-Weyl inner product:

$$\mathrm{BWdist}_{f,X}\colon X \to \mathbb{R}\,,\quad \mathrm{BWdist}_{f,X}(x) \coloneqq \langle f-x, f-x\rangle_{\mathrm{BW}}\,.$$

When f is generic, the number of nonsingular complex critical points of $\mathrm{BWdist}_{f,X}$ is finite and constant, and coincides with the *Euclidean Distance (ED) degree of* X [1] with respect to the Bombieri-Weyl inner product. Furthermore, the nonsingular complex critical points of $\mathrm{BWdist}_{f,X}$ correspond to the so-called *E-eigenvectors of* f [2, 3], generalizing the notion of eigenvector of a symmetric matrix.

We describe the locus of symmetric tensors f having at least one nonsingular critical point $x \in X$ of $\mathrm{BWdist}_{f,X}$ which is k-osculating for some integer $k > 1$, meaning that the gradient of $\mathrm{BWdist}_{f,X}$, when evaluated at x, is orthogonal to the k-th osculating space of X at x. This leads to the notion of *k-osculating eigenvector* of a symmetric tensor. When X is irreducible, then the data locus of symmetric tensors having at least one k-osculating eigenvector is an irreducible variety. We compute its dimension and degree. Furthermore, given a tensor f on this data locus, we study the algebraic scheme of complex k-osculating critical points of $\mathrm{BWdist}_{f,X}$, which is reduced and zero-dimensional for a sufficiently generic f.

This topic belongs to a more general study of *higher-order ED degrees and ED data loci*, a joint work in preparation with Sandra Di Rocco and Kemal Rose.

References

[1] J. Draisma, E. Horobeţ, G. Ottaviani, B. Sturmfels, and R. R. Thomas, The Euclidean distance degree of an algebraic variety, *Found. Comput. Math.* **16** (2016) 99–149.

[2] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, *in 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing 2005*, Piscataway, NJ, 129–132.

[3] L. Qi, Eigenvalues and invariants of tensors, *J. Math. Anal. Appl.* **325** (2007), 1363–1377.