Rigidity results and qualitative properties for solutions of the Lane-Emden equation on Cartan-Hadamard (model) manifolds

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The Cartan-Hadamard conjecture states that on any n-dimensional Cartan-Hadamard manifold $\mathbb{M}^n$ the isoperimetric inequality is satisfied with *Euclidean* optimal constant, and rigidity holds in the sense that equality is attained by some bounded Borel set E if and only if E is isometric to a Euclidean ball (up to negligible sets). So far, the conjecture has been proved to be true for all dimensions $n \leq 4$. Moreover, it has been shown that its validity implies in turn the validity of the *p-Sobolev inequalities* for all $p \in (1, n)$, still with Euclidean optimal constants (see *e.g.* [1]).

In the first part of the talk, I will discuss some results obtained in [2] regarding the classification of Cartan-Hadamard manifolds that support optimal functions for the p-Sobolev inequality, that is, nontrivial functions attaining the best constant of the inequality (*i.e.* Sobolev minimizers). Specifically, we show that under the validity of the Cartan-Hadamard conjecture, the only Cartan-Hadamard manifold for which such optimal functions exist is the Euclidean space $\mathbb{R}^n$ (up to isometries).

In the second part of the talk, I will focus on the classification of *radial solutions* of the corresponding Euler-Lagrange equation, which need not be p-Sobolev optimal functions (this is also contained in [2]); note that the latter turns out to be a semilinear elliptic equation of *Lane-Emden* type. More in general, we study the *critical* or *supercritical* equation

$$-\Delta_p u = -\operatorname{div}\Big(|\nabla u|^{p-2}\,\nabla u\Big) = u^q \quad \text{on } \mathbb{M}^n\,, \qquad u > 0\,,$$

where $q \geq p^* - 1$ and $p^* = \frac{pn}{n-p}$. We prove that, if $\mathbb{M}^n$ is in addition spherically symmetric (*i.e.* it is a *model manifold*) and there exists a radial solution of such an equation that has *finite energy*, then $q = p^* - 1$, $\mathbb{M}^n$ is necessarily isometric to $\mathbb{R}^n$, and u is an Aubin-Talenti function. Furthermore, we show that solutions with infinite energy always exist, providing a detailed description of their asymptotic behavior; in particular, a significant dichotomy happens depending on the *p-stochastic completeness* or *incompleteness* of $\mathbb{M}^n$. A similar analysis, restricted to the case $p = 2$, is also performed for a related Lane-Emden *system* (see [3]), where several technical difficulties arise.

Finally, if time allows, some open problems and some work in progress regarding the *subcritical equation* will be briefly addressed.

The talk is based on joint works with Nicola Soave (Università degli Studi di Torino, Italy).

References

[1] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lect. Notes Math., 1999.

[2] M. Muratori, N. Soave, Some rigidity results for Sobolev inequalities and related PDEs on Cartan-Hadamard manifolds, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **24,** 2 (2023), 751–792.

[3] M. Muratori, N. Soave, The Lane-Emden system on Cartan-Hadamard manifolds: asymptotics and rigidity of radial solutions, *Int. Math. Res. Not. IMRN*, 12 (2024), 9910–9935.