

The space of monoid preorders

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A (total) preorder on the monoid Mon_n of monomials in n variables is a reflexive, transitive, and total relation \leq which is compatible with the monoid structure in the sense that $v \leq w$ implies $u \cdot v \leq u \cdot w$. The preorders that also satisfy the opposite implication are well-known from Gröbner theory, and by a classical result of Robbiano, they can be described by weight matrices. But the space of preorders that do not necessarily satisfy this opposite implication is much wilder, not unlike the Berkovich space of an affine variety is much wilder than any of its tropicalisations. Nevertheless, we have derived satisfactory descriptions of such preorders involving finitely many real numbers, and of the space they form — e.g., it is an irreducible spectral topological space in which every point is open in its closure. We have used these results to settle a decision problem from universal algebra. Roughly speaking, it is decidable whether a boolean combination of inequalities of the form $v \leq w$ (where u, v are products of variables) is a tautology in the theory of totally ordered commutative monoids.