

# Inverse Images of Relative Interiors through Metric Projections in Hilbert Spaces

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## Abstract

In this paper, we study the inverse image and its closure, under the metric projector operator, onto a closed convex set of that convex set and an arbitrary face of it. In an arbitrary real Hilbert space, assuming that the relative interior is nonempty, we characterize the inverse image of the relative interior of the closed convex set, while for a convex subset, we describe its intersection with the given convex set. However, in the finite-dimensional setting, we provide a characterization of the inverse image of a convex subset. Also, for a polyhedral set, we study the inverse images and their closures of the relative interiors of its faces. This characterization is provided for facets (1-codimensional faces) as well as for 2 and 3-codimensional faces, and in fact, our results extend to finite-codimensional faces as well.

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