Necessary Optimality Conditions for Generalized Weak Optimal Solutions in Set-Valued Optimization

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The specialized literature abounds in approaches of tackling the notion of an optimal solution for set-valued optimization problems, of the form:

$$\begin{cases} \min F(x) \\ G(x) \in C \end{cases}$$

wehere $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ are set-valued functions. Among first such attempts we mention the so-called scalar approach, where a certian poin $x_0 \in X$ is said to be an optimal solution, if there exists $y_0 \in F(x_0)$ such that y_0 is a minimal point of the set $\bigcup_{x\in X G(x)\in C} F(x)$. Another approach, the so-called *the natural criteria* in set-valued optimization was first introduced by D. Kuroiwa in 1998 and it basically considers $x_0 \in X$ as an optimal solution, if the set $F(x_0)$ is minimal, when compared the other sets $F(x)$, in terms of a set-inclusion relation, defined by the means of a convex cone.

Continuing Kuroiwa's idea, we introduce a new weak set-relation, with the help of the notion of the quasi-relative interior of a convex cone. In terms of that, we are able to deliver a new optimality notion of a weak solution of a set-valued optimization problems. This notion is accompanied by necessary optimality conditions. As well, we introduce a set-valued dual and a duality theorem. I began this path of research within [2] and [3]

Let A and B belong to $\mathcal{P}_0(Y)$ and $K \subseteq Y$ be a nonempty, pointed convex cone. Then we write:

(a) $A \trianglelefteq^l_{\mathrm{qiK}} B$ if $B \subseteq A + \mathrm{qiK}$ and $\mathrm{A} \trianglelefteq^{\mathrm{u}}_{\mathrm{qiK}} \mathrm{B}$ if $\mathrm{A} \subseteq \mathrm{B} - \mathrm{qiK}$.

(b) an $\mathrm{l} - \mathrm{Min}_{\mathrm{qi}}$**-efficient set** of $\mathcal{S}$, if for each set $B \in \mathcal{S}$ satisfying

$$B \trianglelefteq^l_{\mathrm{qiK}} A, \text{ the relation } A \trianglelefteq^l_{\mathrm{qiK}} B \text{ holds.}$$

(c) an $\mathrm{u} - \mathrm{Min}_{\mathrm{qi}}$**-efficient set** of $\mathcal{S}$, if for each set $B \in \mathcal{S}$ satisfying

$$B \trianglelefteq^u_{\mathrm{qiK}} A, \text{ the relation } A \trianglelefteq^u_{\mathrm{qiK}} B \text{ holds.}$$

An extension of the classical Fenchel theorem can be stated in this case, as it is seen below: let $x_0, x_1 \in \mathrm{domF}$, and let $T \in \mathcal{L}(X, Y)$ be such that $F(x_1) - Tx_1 \in -F^*_{\mathrm{qi_K}}(T)$. Then

(a) If $F(x_0) - Tx_0 \trianglelefteq^l_{\mathrm{qi_K}} F(x_1) - Tx_1$, then $F(x_1) - Tx_1 \trianglelefteq^l_{\mathrm{qi_K}} F(x_0) - Tx_0$.

(b) If $F(x_0) - Tx_0 \trianglelefteq^l_{\mathrm{qi_K}} F(x_1) - Tx_1$, then $F(x_1) - Tx_1 \sim^l F(x_0) - Tx_0$.

References

[1] D. Kuroiwa, The natural criteria in set-valued optimization, *RIMS Kokyuroku* **2049,** (1998) 86–92.

[2] A. Grad, *Generalized Duality and Optimality Conditions*, Mega, 2010.

[3] A. Grad, A generalized interior approach to constrained set-valued duality, *Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity* **12,** (2014), 37–56