Schofield sequences in the Euclidean case

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Consider a finite dimensional quiver algebra kQ over the field k and consider its representations (modules). A pair of indecomposable modules (Y, X) is called an *orthogonal exceptional pair* if the modules are exceptional (no self-extensions) and $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = \mathrm{Ext}^1(X, Y) = 0$. Denote by $\mathcal{F}(X, Y)$ the full subcategory of objects having filtration with factors X and Y. Observe that is $\mathcal{F}(X, Y)$ is an exact, hereditary, abelian subcategory equivalent to the category of finite dimensional k-representations of the quiver having two vertices and $d = \dim_k \mathrm{Ext}^1(Y, X)$ arrows from left to right.

We know due to Hubery that in the euclidean case (when Q is a tame quiver) we have $d = \dim_k \mathrm{Ext}^1(Y, X) \leq 2$, moreover if equality holds then $\underline{\dim}(X \oplus Y) = \delta$ (the minimal radical vector) and the defect $\partial Y = 1$. Thus in the case $\dim_k \mathrm{Ext}^1(Y, X) = 2$ we have $X = P$ indecomposable preprojective of defect -1 and $Y = I$ indecomposable preinjective of dimension $\delta - \underline{\dim}P$ of defect 1. This pair (I, P) is then called *Kronecker pair* since the category $\mathcal{F}(P, I)$ is equivalent to the category of finite dimensional k-representations of the Kronecker quiver K.

The following theorem by Schofield makes possible to construct exceptional modules as extensions of smalle exceptional ones. This procedure is generally called *Schofield induction.*

Theorem 1 *(Schofield, Ringel) If Z is exceptional but not simple, then $Z \in \mathcal{F}(X, Y)$ for some orthogonal exceptional pair (Y, X), and Z is not a simple object in $\mathcal{F}(X, Y)$. In fact, there are precisely $s(Z) - 1$ such pairs, where $s(Z)$ is the support of Z (i.e the number of nonzero components in $\underline{\dim}Z$).*

Note that in the theorem above the condition requiring Z not to be a simple object in $\mathcal{F}(Y, X)$ is equivalent with the existence of an exact sequence of the form

$$0 \longrightarrow uX \longrightarrow Z \longrightarrow vY \longrightarrow 0$$

with u, v positive and uniquely determined by X, Y, Z. Such an exact sequence will be called a *Schofield sequence* and the pair (Y, X) a *Schofield pair* associated to Z.

As Ringel states, whereas it is easy to construct Z given X and Y, there is no convenient procedure yet to determine the possible modules X (called *Schofield submodules* of Z) and then Y, when Z is given. In the Dynkin case Bo Chen's Theorem provides a method to find at least some of these modules X, namely the Gabriel-Roiter submodules of Z.

The talk is focused on some joint work with István Szöllősi. More precisely we describe all the tame Schofield sequences using a numerical criteria for the characterization of these sequences. As a result we give an explicit procedure to obtain all the possible Schofield submodules for a given exceptional module. We will notice that all the tame Schofield sequences are field independent, thus of purely combinatorial nature, depending only on the oriented quiver.