Signatures of integer expansions in real quadratic fields

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Number expansions can be constructed in many different ways. One of the most natural way is to consider a lattice, a linear operator acting on it and a finite digit set describing the expansions. This paper deals with the ring of integers in the real quadratic fields with canonical digit sets and examines the possible signatures of the expansions. The authors present theoretical and algorithmic results as well.

Let $\mathbb{Q}$ be the field of rational numbers, $F \geq 2$ be a square-free integer. It is known that if $F \not\equiv 1 \pmod 4$ then $\{1, \delta\}$, while for $F \equiv 1 \pmod 4$ $\{1, \omega\}$ is an integer basis of $\mathbb{Q}(\sqrt{F})$, spreading the lattices Λ_δ and Λ_ω, respectively, where $\delta = \sqrt{F}$, and $\omega = (1+\sqrt{F})/2$. Let $\alpha = a + b\delta$ or $\alpha = a + b\omega$, $a, b \in \mathbb{Z}$ for which $|\alpha|, |\overline{\alpha}| > 1$, and consider the linear operators

$$M_1 = \begin{pmatrix} a & Fb \\ b & a \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} a & Eb \\ b & a+b \end{pmatrix} \tag{1}$$

acting on the lattices Λ_δ and Λ_ω, respectively, where $E = (F-1)/4$. Let us define the (canonical) set $D = \{je_1\}$ $(j = 0, 1, \ldots, |\alpha\overline{\alpha}| - 1)$, where e_1 is the unit vector. It is known ([1]) that D is a full residue system modulo M $(= M_1$ or $M_2)$ if and only if $(a, b) = 1$ and in this case the linear map $\varphi(z) = M^{-1}(z - d)$ is ultimately (and finitely) periodic, where $z \equiv d \pmod M$, $d \in D$ (see e.g. [3]). In a given system $(\mathbb{Z}^2, M, D)$ we denote the set of periodic points by $\mathcal{P}$. A *signature* $[s_1, s_2, \ldots, s_v]$ of that system is a finite sequence of non-negative integers in which the periodic structure of $\mathcal{P}$ consists of $\#s_i$ cycles with period length i $(1 \leq i \leq v)$. The paper presents the examination of signatures of expansions in the lattices Λ_δ and Λ_ω with operators defined in (1) and with canonical digit sets. Using our theoretical results we present an efficient algorithm for finding the signatures.

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